General Quantum Resonances of the Kicked Particle

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Abstract

The quantum resonances (QRs) of the kicked particle are studied in a most general framework by also considering *arbitrary* periodic kicking potentials. It is shown that QR can arise, in general, for *any rational* value of the Bloch quasimomentum. This is illustrated in the case of the main QRs for arbitrary potentials. In this case, which is shown to be precisely described by the linear kicked rotor, exact formulas are derived for the diffusion coefficients determining the asymptotic evolution of the average kinetic energy of either an incoherent mixture of plane waves or a general wave packet. The momentum probability distribution is exactly calculated and studied for a two-harmonic potential. It clearly exhibits new resonant values of the quasimomentum and it is robust under small deviations from QR.

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I. INTRODUCTION

There has been recently a considerable experimental [1] and theoretical [2,3] interest in new remarkable phenomena associated with the quantum resonances of the periodically kicked particle (KP), either in the presence or in the absence of gravity. In this paper, we report about new general results concerning the quantum resonances of the KP in the absence of gravity. These results extend some of previous work [2,3] in a significant way, shedding light on new basic aspects of the problem and connecting an important case of the system with a well-known model. We start with a brief summary of previous results [2,3], using notation in Ref. [3]. The quantum KP is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + kV(\hat{x})\sum_t \delta(t' - t\tau), \tag{1}$$

where $(x,\ p)$ are the position and momentum of the particle, k is a nonintegrability parameter, $V(x)$ is a periodic potential, t takes all the integer values, t' is the continuous time, and τ is the kicking period. The units are chosen so that the particle mass is 1, the Planck's constant $\hbar = 1$, and the period of $V(x)$ is 2π. In practice, $V(x)$ has been always chosen as the standard potential $V(x) = \cos(x)$. The one-period evolution operator for (1), from $t' = t + 0$ to $t' = t + \tau + 0$, is given by

$$\hat{U} = \exp\left[-ikV(\hat{x})\right]\exp\left(-i\tau\hat{p}^2/2\right). \tag{2}$$

The translational invariance of (2) in $\hat{x}$ implies the conservation of a quasimomentum β $(0 \leq \beta < 1)$: The application of $\hat{U}$ on a Bloch function $\Psi_\beta(x) = \exp(i\beta x)\psi_\beta(x)$, where $\psi_\beta(x + 2\pi) = \psi_\beta(x)$, results in a Bloch function $\Psi'_\beta(x) = \exp(i\beta x)\psi'_\beta(x)$ associated with the same value of β. Here $\psi'_\beta(x)$ is the 2π-periodic function $\psi'_\beta(x) = \hat{U}_\beta\psi_\beta(x)$, where

$$\hat{U}_\beta = \exp\left[-ikV(\hat{x})\right]\exp\left[-i\tau\left(\hat{p} + \beta\right)^2/2\right]. \tag{3}$$

The restriction of the operator (3) to 2π-periodic functions $\psi_\beta(x)$ allows one to interpret x as an angle θ and $\hat{p}$ as an angular-momentum operator $\hat{N} = -id/d\theta$ with integer eigenvalues n. One can then view (3) as the one-period evolution operator for a "β-kicked rotor" (β-KR).

Now, an arbitrary KP wave packet $\Psi(x)$ can be always expressed as a superposition of Bloch functions, $\Psi(x) = \int_0^1 d\beta \exp(i\beta x)\psi_\beta(x)$, where

$$\psi_\beta(x) = \frac{1}{\sqrt{2\pi}} \sum_n \widetilde{\Psi}(n+\beta) \exp(inx), \tag{4}$$

$\widetilde{\Psi}(p)$ being the momentum representation of $\Psi(x)$. One then gets the basic relation

$$\hat{U}^t \Psi(x) = \int_0^1 d\beta \exp(i\beta x) \hat{U}_\beta^t \psi_\beta(x) \tag{5}$$

for integer "time" t, connecting the quantum dynamics of the KP with that of β-KRs.

For typical irrational values of $\tau/(2\pi)$, β-KRs are expected to feature dynamical localization in the angular momentum n [4,5], implying a similar localization of the KP wave packet $\widetilde{\Psi}(p)$ in the momentum $p = n + \beta$ (n and β are, respectively, the integer and fractional parts of p). If $\tau/(2\pi)$ is rational, the usual ($\beta = 0$) KR exhibits quantum resonance (QR), i.e., a *ballistic* (quadratic in time) growth of its kinetic energy [5]. QR in general β-KRs appears to have been studied only in the case of integer $\tau/(2\pi)$ ("main" QRs) with $V(x) = \cos(x)$ [2,3]. It was found [3] that QR arises in this case *only* for special "resonant" values of β, *finite* in number. Thus, QR is exhibited by the KP only if $\widetilde{\Psi}(p)$ is delta localized on the discrete set of momenta $p = n + \beta$, with β in the finite resonant set. However, QR leaves a clear fingerprint in the evolution of either a general KP wave packet (5) or an incoherent mixture of plane waves: In both cases, which involve *all* values of β, the average kinetic energy grows *diffusively* (linearly) in time [2,3]. This diffusive behavior is robust under small deviations ϵ of $\tau/(2\pi)$ from integers, in the sense that it is still observed on time scales $t \propto |k\epsilon|^{-1/2}$ [3].

In this paper, the results above concerning the QRs of the system (1) are extended by also considering *arbitrary* periodic potentials $V(x)$. In Sec. II, we show that the general condition for QR in β-KRs is just the *rationality* of *both* $\tau/(2\pi)$ *and* β. Thus, a resonant value of β may be a *general rational* number in $[0,\ 1)$ and one then has a (countable) infinity of such values. This is illustrated in Sec. III in the case of the main QRs for arbitrary $V(x)$.

We show that this case is precisely described by the well-known linear kicked rotor (linear KR) [9,10]. We focus on this case also in Secs. IV and V. In Sec. IV, we derive exact formulas for the diffusion coefficients determining the asymptotic evolution of the average kinetic energy of either an incoherent mixture of plane waves or a general KP wave packet. For a mixture uniformly distributed in β, the formula essentially coincides with one given by Berry in the context of the linear KR [10]. Multi-harmonic potentials cause the diffusion coefficient for a general wave packet to be explicitly dependent on quantum correlations reflecting the localization features of the initial wave packet in momentum space. In Sec. V, the momentum probability distribution is exactly calculated and studied for a two-harmonic potential. It clearly exhibits new resonant values of β due to the second harmonics and it is found numerically to be robust under small deviations of $\tau/(2\pi)$ from integers. A summary and conclusions are presented in Sec. VI.

II. GENERAL QR CONDITIONS

The basic origin of QR is a band quasienergy (QE) spectrum due to some translational invariance in phase space. For KR systems, this is the invariance of the evolution operator under translations $\hat{T}_q = \exp(-iq\hat{\theta})$ by q in the angular momentum $\hat{N}$ [5,6]; here q must be an integer since θ is an angle ($0 \leq \theta < 2\pi$). The translational invariance of β-KRs is expressed by $[\hat{U}_\beta,\ \hat{T}_q] = 0$, where $\hat{U}_\beta$ is given by Eq. (3) ($\hat{x} \rightarrow \hat{\theta}$, $\hat{p} \rightarrow \hat{N}$). Using the fact that $\hat{N}$ has integer eigenvalues, one easily finds that $[\hat{U}_\beta,\ \hat{T}_q] = 0$ is satisfied only if

$$\frac{\tau}{2\pi} = \frac{l}{q}, \tag{6}$$

$$\beta = \frac{r}{l} - \frac{q}{2} \bmod(1), \tag{7}$$

where l and r are integers. Eq. (6) is the rationality condition for $\tau/(2\pi)$ while Eq. (7) is a formula for the general resonant values of β. For definiteness and without loss of generality, we assume that l and q are positive. Let us now write $l = gl_0$ and $q = gq_0$, where l_0 and q_0 are coprime positive integers and g is the greatest common factor of $(l,\ q)$; the value of

$\tau/(2\pi) = l_0/q_0$ will be kept fixed in what follows. It is then clear that β in Eq. (7) can take *any rational* value $\beta_{\rm r}$ in $[0,\ 1)$ since g can be always chosen so that $r = (\beta_{\rm r} + gq_0/2)gl_0$ is integer. For given $\beta = \beta_{\rm r}$, we shall choose g as the *smallest* positive integer satisfying the latter requirement, so as to yield the minimal values of $l = gl_0$ and $q = gq_0$. In general, $g > 1$, so that $(l,\ q)$ are *not* coprime. For the usual KR ($\beta = 0$), $g = 1$ if l_0q_0 is even and $g = 2$ if l_0q_0 is odd (compare with Ref. [6]). We denote $\beta_{\rm r}$ by $\beta_{r,g}$, where the integer $r = (\beta_{\rm r} + gq_0/2)gl_0$ labels all the different values of $\beta_{\rm r}$ for given minimal g.

The QE states φ for $\beta = \beta_{r,g}$ can be chosen as simultaneous eigenstates of $\hat{U}_\beta$ and $\hat{T}_q$: $\hat{U}_\beta\varphi = \exp(-i\omega)\varphi$, $\hat{T}_q\varphi = \exp(-iq\alpha)\varphi$, where ω is the QE and α is a "quasiangle", varying in the "Brillouin zone" (BZ) $0 \leq \alpha < 2\pi/q$. One may view the Bloch function $\exp(i\beta x)\varphi(x)$ as a state on the "quantum torus" $0 \leq x < 2\pi$, $0 \leq p < q$, with toral boundary conditions [7] specified by $(\alpha,\ \beta)$. Using standard methods [5,6], it is easy to show from the eigenvalue equations above that at fixed α one has precisely q QE eigenvalues $\omega_b(\alpha,\ \beta)$, $b = 0, \ldots,\ q-1$. Since $q = gq_0$ is minimal, the BZ is maximal for the given value of $\beta = \beta_{r,g}$. Then, as α is varied continuously in the BZ, the q eigenvalues form q QE bands. These bands are expected to be, typically, not all flat (with zero width); QR can then arise and $\beta = \beta_{r,g}$ is indeed a resonant value. In the nontypical case that all the bands are flat, $\beta = \beta_{r,g}$ is nonresonant: QR is replaced by a bounded quantum motion, the "quantum antiresonance" [8].

III. CASE OF MAIN QRs: CONNECTION WITH THE LINEAR KR

Previous studies [2,3] have focused on the important case of the main QRs ($\tau = 2\pi l_0$, $q_0 = 1$), assuming that $V(\theta) = \cos(\theta)$. It was found [3] that only l_0 values of β are resonant (exhibit QR). They are given by Eq. (7) with $r = 0,\ 1, \ldots,\ l_0 - 1$ and $g = 1$ (i.e., $l = l_0$ and $q = 1$). In this section, we study the main QRs for arbitrary $V(\theta)$ and we show that all rational values of β in $[0,\ 1)$ are resonant if $V(\theta)$ contains all the harmonics. The case of $\tau = 2\pi l_0$ is the only one in which the term $(\hat{N} + \beta)^2$ in Eq. (3) ($\hat{x} \rightarrow \hat{\theta}$, $\hat{p} \rightarrow \hat{N}$) can be replaced by the operator $\hat{N} + 2\beta\hat{N} + \beta^2$, *linear* in $\hat{N}$; this is because

$\exp(-i\pi l_0 n^2) = \exp(-i\pi l_0 n)$ for the integer eigenvalues n of $\hat{N}$. Then, after omitting the nonrelevant constant phase factor $\exp(-i\pi l_0 \beta^2)$, one can express (3) as follows:

$$\hat{U}_\beta = \exp\left[-ikV(\hat{\theta})\right] \exp\left(-i\tau_\beta \hat{N}\right), \tag{8}$$

where $\tau_\beta = \pi l_0(2\beta + 1)$. We identify $\hat{U}_\beta$ in Eq. (8) as the one-period evolution operator for the well-known linear KR [9,10] with Hamiltonian $\hat{H} = \tau_\beta \hat{N} + kV(\hat{\theta}) \sum_{t=-\infty}^{\infty} \delta(t' - t)$; the corresponding Schrödinger equation is exactly solvable for arbitrary potential $V(\theta)$,

$$V(\theta) = \sum_m V_m \exp(-im\theta). \tag{9}$$

Assuming the quantum state of the linear KR to be initially (at $t = 0$) an angular-momentum state, $\psi_{\beta,0}(\theta) = \exp(in_0\theta)/\sqrt{2\pi}$, the expectation value of the kinetic energy at time t is given by the exact expression [10,11]:

$$E_{n_0,\beta}(t) = \frac{1}{2}\left\langle \psi_{\beta,t}|\hat{N}^2|\psi_{\beta,t}\right\rangle = \frac{n_0^2}{2} + k^2 \sum_{m>0} m^2 |V_m|^2 \frac{\sin^2(m\tau_\beta t/2)}{\sin^2(m\tau_\beta/2)}. \tag{10}$$

If β is a typical irrational number, so that also $\tau_\beta/(2\pi) = l_0(\beta + 1/2)$ is such, the QE spectrum of the linear KR is pure point [9] and $E_{n_0,\beta}(t)$ in Eq. (10) is a bounded, quasiperiodic function of time [10]. Consider now a general rational value of $\beta = \beta_{r,g}$, given by Eq. (7) with $l = gl_0$, $q = gq_0 = g$, and g minimal (see Sec. II). It is easy to show that $\beta_{r,g}$ can be written, up to a fixed integer shift in r by $gl_0/2$ if both g and $gl_0/2$ are even, as $\beta_{r,g} = r/(gl_0) - 1/2 \bmod(1)$, where r and g are coprime; the corresponding value of $\tau_\beta/(2\pi)$ is r/g, up to some additive integer. Then, from work [9], the QE spectrum consists of g bands which are all nonflat (i.e., $\beta = \beta_{r,g}$ is resonant) if there exists at least one integer $j \neq 0$ such that the Fourier coefficient V_{jg} in Eq. (9) is nonzero; otherwise, all the g bands are flat ($\beta = \beta_{r,g}$ is nonresonant, with QR replaced by quantum antiresonance [8]). In fact, if $V_{jg} \neq 0$ for some $j \neq 0$, one finds from Eq. (10) (with $\tau_\beta = 2\pi r/g$) a ballistic behavior for large t, $E_{n_0,\beta}(t) \approx St^2/2$, where $S = k^2 g^2 \sum_j j^2 |V_{jg}|^2$ [10]. Thus, if $V(\theta)$ contains all the harmonics ($V_m \neq 0$ for all m), all rational values $\beta_{r,g}$ of β are resonant.

As a simple example, let $V(\theta) = \cos(\theta) + \gamma\cos(2\theta)$. The only nonzero coefficients V_m are $V_{\pm 1} = 1/2$ and $V_{\pm 2} = \gamma/2$, so that QR arises only for $g = 1,\ 2$. The resonant β values are $\beta_{r,g} = r/(gl_0) - 1/2 \bmod(1)$, where $r = 0,\ 1, \ldots,\ l_0 - 1$ for $g = 1$ and $r = 1,\ 3, \ldots,\ 2l_0 - 1$ for $g = 2$. The values for $g = 1$ are just the known ones for $V(\theta) = \cos(\theta)$ [3] while those for $g = 2$ are new ones, due entirely to the second harmonics. See, however, note [12].

IV. ASYMPTOTIC DIFFUSION OF AVERAGE KINETIC ENERGY

A. Incoherent Mixture of Plane Waves

Let us assume, as in experimental situations [1], that the initial KP state is an incoherent mixture of plane waves $\exp(ipx)$ with momentum distribution $f(p)$ sufficiently localized in p. By decomposing p into its integer and fractional parts, $p = n + \beta$, the average kinetic energy of this mixture at time t can be expressed as $\bar{E}(t) = \int_0^1 d\beta \sum_n f(n+\beta) E'_{n,\beta}(t)$; here $E'_{n,\beta}(t)$ is the expectation value of the kinetic energy in the state evolving from a plane wave. In the case of $\tau = 2\pi l_0$, on which we shall focus, $E'_{n,\beta}(t)$ is given by the right-hand side of Eq. (10) with n_0 replaced by $p = n + \beta$; this can be easily seen from Eqs. (18)-(21) in Ref. [10]. As in Ref. [3], we define $f_0(\beta) = \sum_n f(n+\beta)$ and use the asymptotic (large t) relation

$$\int_0^1 d\beta f_0(\beta) \frac{\sin^2\left[\pi m l_0 \left(\beta + 1/2\right) t\right]}{\sin^2\left[\pi m l_0 \left(\beta + 1/2\right)\right]} \sim \frac{t}{|m| l_0} \sum_{r=0}^{|m|l_0 - 1} f_0\left(\beta_{r,m}\right),$$

where $\beta_{r,m} = r/(|m|l_0) - 1/2 \bmod(1)$. We then find that $\bar{E}(t)$ behaves diffusively for large t, $\bar{E}(t) \sim D_0 t$, where the diffusion coefficient D_0 is given by

$$D_0 = \frac{k^2}{l_0} \sum_{m>0} m|V_m|^2 \sum_{r=0}^{ml_0 - 1} f_0\left(\beta_{r,m}\right). \tag{11}$$

The simple mixture with $f(n+\beta) = \delta_{n,n_0}$ is uniformly distributed in β, $f_0(\beta) = 1$. In this case, $\bar{E}(t) = \int_0^1 d\beta E'_{n_0,\beta}(t)$, leading to an exact equality for all times t: $\bar{E}(t) = \bar{E}(0) + D_0 t$ with $D_0 = k^2/(4\pi) \int_0^{2\pi} [dV(\theta)/d\theta]^2 d\theta$. The latter result was essentially obtained by Berry [10] as the kinetic energy of a linear KR whose value of τ is completely unknown.

B. General Wave Packet

Next, we consider a general KP wave packet (5), a "coherent mixture" of Bloch waves exhibiting all values of β. The expectation value of the kinetic energy in $\Psi_t(x) = \hat{U}^t\Psi(x)$ is

$$\langle E\rangle_t = \frac{1}{2}\left\langle \Psi_t\left|\hat{p}^2\right|\Psi_t\right\rangle \sim \frac{1}{2}\int_0^1 d\beta \int_0^{2\pi} d\theta \left|\frac{d\psi_{\beta,t}(\theta)}{d\theta}\right|^2, \tag{12}$$

where the last relation, with $\psi_{\beta,t}(\theta) \equiv \hat{U}_\beta^t\psi_\beta(\theta)$, holds for large t provided that $\langle E\rangle_t$ is unbounded as $t \to \infty$ [2]. In fact, we now show that $\langle E\rangle_t$ exhibits an asymptotic diffusive behavior for $\tau = 2\pi l_0$. In this case, we easily obtain from Rels. (8) and (9) that

$$\psi_{\beta,t}(\theta) \equiv \hat{U}_\beta^t\psi_\beta(\theta) = \exp\left[-ik\bar{V}_{\beta,t}(\theta)\right]\psi_\beta(\theta - t\tau_\beta), \tag{13}$$

where

$$\bar{V}_{\beta,t}(\theta) = \sum_{s=0}^{t-1} V\left(\theta - s\tau_\beta\right) = \sum_m V_m \frac{\sin(m\tau_\beta t/2)}{\sin(m\tau_\beta/2)} e^{-im[\theta-(t-1)\tau_\beta/2]}. \tag{14}$$

As shown in Ref. [2] (Appendix A) for $V(\theta) = \cos(\theta)$, with straightforward extension to arbitrary $V(\theta)$, the expression (13) implies that for large t the dominant contribution of $|d\psi_{\beta,t}(\theta)/d\theta|^2$ to $\langle E\rangle_t$ in Eq. (12) is $k^2\left|\psi_\beta(\theta - t\tau_\beta)d\bar{V}_{\beta,t}(\theta)/d\theta\right|^2$. This contribution can be calculated using Eq. (14) and a relation following from Eq. (4):

$$|\psi_\beta(\theta)|^2 = \frac{1}{2\pi}\sum_m C_\beta(m)\exp(im\theta), \tag{15}$$

where $C_\beta(m)$ are correlations in momentum space,

$$C_\beta(m) = \sum_n \widetilde{\Psi}(m+n+\beta)\widetilde{\Psi}^*(n+\beta). \tag{16}$$

From Eqs. (14) and (15), one can write the Fourier expansion of $k^2\left|\psi_\beta(\theta - t\tau_\beta)d\bar{V}_{\beta,t}(\theta)/d\theta\right|^2$. After inserting this expansion in Eq. (12), we get

$$\langle E\rangle_t \sim \frac{k^2}{2}\sum_{m,m'\neq 0} mm'V_m V_{m'}^* B(m,\ m';\ t), \tag{17}$$

where

$$B(m,\ m';\ t)=\int_0^1 d\beta\frac{\sin(m\tau_\beta t/2)}{\sin(m\tau_\beta/2)}\frac{\sin(m'\tau_\beta t/2)}{\sin(m'\tau_\beta/2)}C_\beta(m-m')e^{i(m'-m)(t+1)\tau_\beta/2}. \tag{18}$$

We show in the Appendix that the asymptotic behavior of the quantity (18) for $t\to\infty$ is given by

$$B(m,\ m';\ t)\sim\frac{t}{2mm'l_0}\left(|m|+|m'|-|m-m'|\right)\sum_{r=0}^{g(m,m')l_0-1}C_{\beta_{r,g}}(m-m'), \tag{19}$$

where $g=g(m,\ m')$ is the greatest common factor of $(|m|,\ |m'|)$ and $\beta_{r,g}=r/(gl_0)-1/2$ mod(1), $r=0,\ldots,\ gl_0-1$. Rels. (17) and (19) imply a diffusive behavior of $\langle E\rangle_t$ for large t, $\langle E\rangle_t\sim Dt$. The diffusion coefficient D can be expressed, after some algebra, as the sum of two terms:

$$\begin{aligned}D=D_{\mathrm{I}}+D_{\mathrm{II}}=&\frac{k^2}{l_0}\sum_{m>0}m|V_m|^2\sum_{r=0}^{ml_0-1}C_{\beta_{r,m}}(0)\\&+\frac{2k^2}{l_0}\sum_{m=1}^{\infty}\sum_{m'=m+1}^{\infty}m\mathrm{Re}\left(V_mV_{m'}^*\right)\sum_{r=0}^{g(m,m')l_0-1}\mathrm{Re}\left[C_{\beta_{r,g}}(m-m')\right].\end{aligned} \tag{20}$$

For a potential containing harmonics V_m of sufficiently high order m, the term D_{II} [on the second line of Eq. (20)] can generally lead to a sensitive dependence of D on the correlations $C_\beta(m)$, which reflect the profile of the initial wave packet in momentum space by Eq. (16). This term vanishes in some cases, e.g., for $V(\theta)=\cos(\theta)$ and/or a uniform probability distribution $|\psi_\beta(\theta)|^2$ in Eq. (15). One is then left with only the first term (D_{I}), which is completely analogous to the diffusion coefficient (11) for the incoherent mixture; this is because $C_\beta(0)=\sum_n\left|\widetilde{\Psi}(n+\beta)\right|^2$ from Eq. (16) and this is analogous to $f_0(\beta)=\sum_n f(n+\beta)$.

V. MOMENTUM PROBABILITY DISTRIBUTION

The momentum probability distribution (MPD) for a KP wave packet is given by $P(p,\ t)=\left|\widetilde{\Psi}_t(p)\right|^2$, where $\widetilde{\Psi}_t(p)$ is the momentum representation of the wave packet at time t. At fixed β, $P(n+\beta,\ t)$ is the angular-momentum (n) distribution for a β-KR. Under QR conditions and for resonant β, the motion of the β-KR is ballistic in n, so that the width $\Delta n(t)$ of $P(n+\beta,\ t)$ increases much faster than that for a nonresonant β [for most values of

β, $\Delta n(t)$ is expected to be essentially bounded due to dynamical localization]. Thus, if p and t are sufficiently large, $P(p,\ t)$ is almost zero except of narrow peaks around $p = n+\beta_{r,g}$, for all resonant values $\beta_{r,g}$ of β. The diffusion coefficient (20) is the average of $P(p,\ t)p^2/(2t)$ over p in the limit of $t \to \infty$; the only nonzero contributions to this average come from the resonant peaks. The contribution of all the peaks with fixed $\beta_{r,g} = \beta_{r',g'}$ is precisely the contribution of all the terms with $\beta_{r,m} = \beta_{r',g'}$ and/or $\beta_{r,g} = \beta_{r',g'}$ in formula (20).

We now study the MPD for $\tau = 2\pi l_0$ and $V(x) = \cos(x) + \gamma\cos(2x)$ (see also end of Sec. III). From Rel. (4), $\widetilde{\Psi}_t(n+\beta)$ are just the Fourier coefficients of $\sqrt{2\pi}\psi_{\beta,t}(\theta)$ and, for $\tau = 2\pi l_0$, $\psi_{\beta,t}(\theta)$ can be calculated using Rel. (13). We assume an initial wave packet $\widetilde{\Psi}(p)$ satisfying $\widetilde{\Psi}(p) = 1$ for $0 \leq p < 1$ and $\widetilde{\Psi}(p) = 0$ otherwise. This corresponds, by Rel. (4), to a uniform $\psi_\beta(\theta)$, $\psi_\beta(\theta) = (2\pi)^{-1/2}$ for all β. Rel. (13) then implies that $\sqrt{2\pi}\psi_{\beta,t}(\theta) = \exp\left[-ik\bar{V}_{\beta,t}(\theta)\right]$. Since only terms with $|m| = 1,\ 2$ appear in the sum (14), the Fourier coefficients of $\exp\left[-ik\bar{V}_{\beta,t}(\theta)\right]$ can be easily expressed, essentially, as a convolution of ordinary Bessel functions $J_n(\cdot)$ in the index n. We finally obtain the exact expression

$$P(p = n+\beta,\ t) = \left| \sum_{m=-\infty}^{\infty} i^m J_{n-2m}\left[k_1(\beta,\ t)\right] J_m\left[k_2(\beta,\ t)\gamma\right] \right|^2, \tag{21}$$

where $k_j(\beta,\ t) = k\sin(j\tau_\beta t/2)/\sin(j\tau_\beta/2)$, $j = 1,\ 2$, and $\tau_\beta = \pi l_0(2\beta+1)$. Let us consider some behaviors of (21) for $\beta = \beta_{r,g}$, where $\beta_{r,g}$ ($g = 1,\ 2$) are the resonant values of β determined in Sec. III: (a) For $\beta_{r,1} = r/l_0 - 1/2 \mod(1)$, $r = 0,\ 1, \ldots,\ l_0 - 1$, one has $|k_1(\beta_{r,1},\ t)| = |k_2(\beta_{r,1},\ t)| = kt$. Now, the Bessel function $J_m(x)$ is relatively small for $|m| > |x|$ and, for $|x| < 1$, $J_0(x) = O(1)$ [13]. We then see that on the time scale $t \lesssim T_\gamma = |k\gamma|^{-1}$ one can approximate (21) at $p = n+\beta_{r,1}$ by $P(p,\ t) \approx J_n^2(kt)$. Thus, the most prominent $g = 1$ peaks of the MPD for $t \lesssim T_\gamma$ are those with $|n| \lesssim |kt|$. As $\gamma \to 0$ ($T_\gamma \to \infty$), the expression (21) reduces exactly to $P(p,\ t) = J_n^2\left[k_1(\beta,\ t)\right]$ (see note [14]). (b) For $\beta_{r,2} = r/(2l_0) - 1/2 \mod(1)$, $r = 1,\ 3, \ldots,\ 2l_0 - 1$, one has $|k_1(\beta_{r,2},\ t)| = k$ or 0 for t odd or even, respectively, and $|k_2(\beta_{r,2},\ t)| = kt$. Then, if $|k| \lesssim 1$, $J_{n-2m}\left[k_1(\beta_{r,2},\ t)\right]$ is relatively small for $|n-2m| > 1$ and one can approximate (21) at $p = n+\beta_{r,2}$ by $P(p,\ t) \approx J_{[n/2]}^2(k\gamma t)$,

where $[n/2]$ is the integer part of $n/2$. The new ($g = 2$) peaks thus start to emerge in a significant way when $t > |k\gamma|^{-1}$ and, for $|n| \lesssim 2|k\gamma t|$, their magnitude should be comparable to that of the $g = 1$ ones. All these behaviors are illustrated in Fig. 1 for $\tau = 2\pi$ ($l_0 = 1$), $t = 100$, $k = 0.1$, $\gamma = 0$ [Fig. 1(a)] and $\gamma = 0.2$ [Fig. 1(b), with $2|k\gamma t| = 4$]. We have checked numerically for many values of $(t,\ k,\ \gamma)$ that the MPD is robust under sufficiently small perturbations of τ, $\tau = 2\pi + \epsilon$, at least within the domains of p where the principal peaks above are found. As an example, compare Fig. 1(c) with Fig. 1(b).

VI. SUMMARY AND CONCLUSIONS

In conclusion, the results in this paper should provide new insights into the nature of the spectra and quantum dynamics of the KP under general QR conditions and for arbitrary potentials. As a direct consequence of translational invariance, any rational value of the Bloch quasimomentum β may be resonant (exhibits QR). At fixed $\tau/(2\pi) = l_0/q_0$ (l_0 and q_0 are coprime integers), a rational β is characterized by an integer pair $(r,\ g)$ which determines β through Eq. (7), with $l = gl_0$ and $q = gq_0$ assuming their minimal values. The QE spectrum consists of $q = gq_0$ bands which should be, typically, not all flat, implying QR. By slightly varying β on the rationals, the corresponding value of g changes erratically, leading to similar changes in the QE spectrum. The important case of the main QRs ($q_0 = 1$) is precisely described by the linear KR and is thus exactly solvable. In this case, all rational values of β are indeed resonant for generic potentials containing all the harmonics. In general, the resonant values of β correspond to peaks in the momentum probability distribution (see Fig. 1) and appear explicitly in the formulas (11) and (20) for the diffusion coefficients. Formula (20) implies a new phenomenon for multi-harmonic potentials [thus excluding the standard case of $V(x) = \cos(x)$]: A sensitive dependence of the diffusion coefficient on the specific localization features of the initial wave packet in momentum space. Assuming the robustness of our results under small variations of τ, which was numerically verified for a two-harmonic potential, it may be possible to observe this phenomenon in experimental realizations of the system. While high-order ($q_0 > 1$) QRs appear to be presently beyond

experimental observation, an interesting question is whether cases of such QRs are exactly solvable, at least to some extent, for the β-dependent QE spectra and quantum dynamics. We do not have yet a definite answer to this question. We hope that we shall be able to make some progress in this direction in future works.

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APPENDIX

We derive here the asymptotic behavior (19). The dominant contributions to the integral (18) come from small β-intervals around the zeros of the denominator of the integrand. We show below that the behavior (19) is completely due to the simultaneous zeros of $\sin(m\tau_\beta/2)$ and $\sin(m'\tau_\beta/2)$ ($m,\ m' \neq 0$), $\tau_\beta = \pi l_0(2\beta+1)$. The zeros of, say, $\sin(m\tau_\beta/2)$ are $\beta = \beta_{r,m} = r/(|m|l_0) - 1/2 \bmod(1)$, $r = 0, \ldots,\ |m|l_0 - 1$. Writing $m = gm_0$ and $m' = gm'_0$, where m_0 and m'_0 are coprime integers and $g = g(m,\ m')$ is the greatest common factor of $(|m|,\ |m'|\ )$, it is easy to see that there are precisely gl_0 simultaneous zeros, given by $\beta = \beta_{r,g} = r/(gl_0) - 1/2 \bmod(1)$, $r = 0, \ldots,\ gl_0 - 1$.

Thus, let $\bar{\beta} = \beta_{r,m}$ be a zero of $\sin(m\tau_\beta/2)$ which is not a zero of $\sin(m'\tau_\beta/2)$, $|m'| \neq |m|$, and consider a β-interval $I_\epsilon = [\bar{\beta} - \epsilon,\ \bar{\beta} + \epsilon]$ sufficiently small that no zero of $\sin(m'\tau_\beta/2)$ lies within it. We show that the contribution of I_ϵ to (18) is finite in the limit of $t \to \infty$. Let $|m|\pi l_0\epsilon \ll 1$, so that $\sin(m\tau_\beta/2) \approx (-1)^r m\pi l_0(\beta - \bar{\beta})$ in I_ϵ. We assume that the correlation function (16) can be expanded as a Taylor series around $\beta = \bar{\beta}$: $C_\beta(m) = C_{\bar{\beta}}(m) + C'_{\bar{\beta}}(m)(\beta - \bar{\beta}) + \cdots$. The dominant contribution of I_ϵ to (18) is then approximately given by

$$B(I_\epsilon) \approx \frac{(-1)^r C_{\bar\beta}(m-m')}{m\pi l_0} \int_{\bar\beta-\epsilon}^{\bar\beta+\epsilon} d\beta \frac{\sin(m\tau_\beta t/2)}{\beta-\bar\beta} \frac{\sin(m'\tau_\beta t/2)}{\sin(m'\tau_\beta/2)} e^{i(m'-m)(t+1)\tau_\beta/2}. \tag{22}$$

We introduce the variable $z = \pi l_0(\beta - \bar\beta)t$ and define $\bar z = \pi l_0(\bar\beta + 1/2)t$. For $t \gg \epsilon^{-1}$, one can see that $B(I_\epsilon)$ in Eq. (22) is well approximated by

$$B(I_\epsilon) \approx \frac{(-1)^{r(t+1)} C_{\bar\beta}(m-m')}{m\pi l_0 \sin(m'\tau_{\bar\beta}/2)} \int_{-\infty}^{\infty} \frac{dz}{z} \sin(mz) \sin\left[m'(z+\bar z)\right] e^{i(m'-m)(z+\bar z)}. \tag{23}$$

The integral in Eq. (23) can be calculated exactly using simple trigonometry and formulas (3.741.2) and (3.763.2) in Ref. [13]; its value is finite for all $\bar z$ (or t).

Let us therefore consider a simultaneous zero $\beta = \beta_{r,g}$ of $\sin(m\tau_\beta/2)$ and $\sin(m'\tau_\beta/2)$, denoting it again by $\bar\beta$. The interval I_ϵ is defined as above with $\pi l_0 \epsilon \ll \min\left(|m|^{-1},\ |m'|^{-1}\right)$. The contribution of I_ϵ to (18) is, approximately,

$$B(I_\epsilon) \approx \frac{(-1)^{r(m_0+m_0')}}{mm'\pi^2 l_0^2} \int_{\bar\beta-\epsilon}^{\bar\beta+\epsilon} d\beta \frac{\sin(m\tau_\beta t/2)\sin(m'\tau_\beta t/2)}{\left(\beta-\bar\beta\right)^2} C_\beta(m-m') e^{i(m'-m)(t+1)\tau_\beta/2}, \tag{24}$$

where m_0 and m_0' are the integers defined above. Expanding again $C_\beta(m-m')$ as a Taylor series around $\beta = \bar\beta$, we see that the first-order term $C'_{\bar\beta}(m-m')(\beta-\bar\beta)$ in this expansion gives a contribution to (24) which is similar to (22) and is thus finite for all t. The contributions of higher-order terms are, obviously, also finite for all t. After simple algebra, one can easily verify that the contribution $B^{(0)}(I_\epsilon)$ of the zero-order term $C_{\bar\beta}(m-m')$ is well approximated, for $t \gg \epsilon^{-1}$, by

$$B^{(0)}(I_\epsilon) \approx \frac{C_{\bar\beta}(m-m')t}{mm'\pi l_0} \int_{-\infty}^{\infty} dz\, z^{-2} \sin(mz)\sin(m'z)\cos\left[(m-m')z\right], \tag{25}$$

where the variable z was defined above. By expressing the product $\sin(m'z)\cos\left[(m-m')z\right]$ as a sum, the integral in Eq. (25) can be calculated exactly using formula (3.741.3) in Ref. [13]. The final result, with $\bar\beta = \beta_{r,g}$, is

$$B^{(0)}(I_\epsilon) \approx \frac{t}{2mm'l_0} \left(|m| + |m'| - |m-m'|\right) C_{\beta_{r,g}}(m-m'). \tag{26}$$

Summing (26) over all the gl_0 simultaneous zeros $\beta = \beta_{r,g}$ of $\sin(m\tau_\beta/2)$ and $\sin(m'\tau_\beta/2)$, we obtain the asymptotic behavior (19).

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[11] In Ref. [10], the potential (9) was assumed to be even (real $V_m = V_{-m}$). We present here the straightforward extension of some results in Ref. [10] to the most general $V(\theta)$.

[12] If $\tau/(2\pi)$ is rational but noninteger ($q_0 > 1$), resonant values of $\beta = \beta_{r,g}$ with $g > 1$ may exist already for $V(\theta) = \cos(\theta)$. For example, in the case of $\tau/(2\pi) = 1/2$ and $V(\theta) = \cos(\theta)$, we found numerically that for $\beta = 1/2$ [minimal $g = 2$, $r = 1$, and $l = gl_0 = 2$ in Eq. (7)] all the $q = gq_0 = 4$ QE bands have nonzero width, implying QR.

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[14] Due to the uniformity of $\psi_\beta(\theta)$, Eq. (21) gives also the angular-momentum distribution for a β-KR initially in the angular-momentum state $n = 0$. Then, the $\gamma = 0$ expression $P(n + \beta,\ t) = J_n^2\,[k_1(\beta,\ t)]$ is just the known [3] exact result for this distribution in the case of $V(\theta) = \cos(\theta)$.

FIGURES

FIG. 1. Momentum probability distribution (21) for $t = 100$, $k = 0.1$, and (a) $\tau = 2\pi$, $\gamma = 0$; (b) $\tau = 2\pi$, $\gamma = 0.2$; (c) $\tau = 2\pi + \epsilon$ [$\epsilon = \pi(\sqrt{5} - 1)/1200 \approx 0.0032$], $\gamma = 0.2$. All the resonant peaks for $\gamma = 0$ [in (a)] are at $p = n + 1/2$. For $\gamma = 0.2$ [in (b) and (c)], new peaks appear at integer $p = n$.

Figure 1